

# Exotic symmetry and monodromy equivalence in Schrödinger sigma models

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## Abstract

We consider the classical integrable structure of two-dimensional non-linear sigma models with target space three-dimensional Schrödinger spacetimes. There are the two descriptions to describe the classical dynamics: 1) the left description based on  $SL(2, \mathbb{R})_L$  and 2) the right description based on  $U(1)_R$ . We have shown the  $sl(2, \mathbb{R})_L$  Yangian and  $q$ -deformed Poincaré algebras associated with them. We proceed to argue an infinite-dimensional extension of the  $q$ -deformed Poincaré algebra. The corresponding charges are constructed by using a non-local map from the flat conserved currents related to the Yangian. The *exotic* tower structure of the charges is revealed by directly computing the classical Poisson brackets. Then the monodromy matrices in both descriptions are shown to be gauge-equivalent via the relation between the spectral parameters. We also give a simple Riemann sphere interpretation of this equivalence.

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# 1 Introduction

In the recent studies of the AdS/CFT correspondence [1–3], Schrödinger spacetimes [4, 5] have been in the spotlight. The spacetimes are basically null-deformations of the usual AdS spaces and the relativistic conformal algebra is broken to a subalgebra called the Schrödinger algebra [6, 7]. Those are considered to be holographic dual to non-relativistic (NR) CFTs [8, 9] realized by using ultra cold atoms in real laboratories. This observation is of great significance because it might be possible to test string theory from tabletop experiments (for reviews see [10, 11]).

To elaborate the AdS/NRCFT correspondence, it is interesting to explore the integrability of two-dimensional non-linear sigma models with target space Schrödinger spacetimes. The integrability in AdS/CFT has played an important role in studying the matching of the spectra (for a comprehensive review see [12]). The integrability yields a powerful tool to examine AdS/NRCFT as well.

Schrödinger spacetimes are represented by non-symmetric and actually non-reductive cosets [13] and so the classical integrable structure is far from obvious<sup>1</sup>. Although the integrability has not been clarified in more than three dimensions so far, the sigma models on three-dimensional Schrödinger spacetimes are shown to be classically integrable by explicitly constructing an infinite set of non-local charges and showing that the classical  $r/s$ -matrices satisfy the extended Yang-Baxter equation [20]<sup>2</sup>. Thus the spacetimes are regarded as *integrable* deformations of  $\text{AdS}_3$ . An intensive analysis on other integrable deformations has been done in the recent work [22].

In this paper we proceed our previous analysis on the classical integrable structure of two-dimensional non-linear sigma models on three-dimensional Schrödinger spacetimes [20]. There are the two descriptions to describe the classical dynamics: 1) the left description based on  $SL(2, \mathbb{R})_L$  and 2) the right description based on  $U(1)_R$ . We have shown the  $sl(2, \mathbb{R})_L$  Yangian and  $q$ -deformed Poincaré algebras associated with them [20]. However, while the Yangian is an infinite-dimensional symmetry, the  $q$ -deformed Poincaré symmetry is finite-dimensional.

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<sup>1</sup>It is well known that symmetric coset sigma models in two dimensions are always classically integrable. For classic works at an early stage of development, see [14–18]. For a comprehensive book, see [19].

<sup>2</sup> For an earlier argument on the construction of conserved charges based on T-duality, see [21].

The main purpose here is to argue an infinite-dimensional extension of the  $q$ -deformed Poincaré algebra. The corresponding charges are constructed by using a non-local map from the flat conserved currents related to the Yangian. The *exotic* tower structure of the charges is revealed by directly computing the classical Poisson brackets. Then the monodromy matrices in both descriptions are shown to be gauge-equivalent via the relation between the spectral parameters. We also give a simple Riemann sphere interpretation of this equivalence.

This paper is organized as follows. In section 2 we introduce the classical action of two-dimensional non-linear sigma models on three-dimensional Schrödinger spacetimes. In section 3 the left description based on  $SL(2, \mathbb{R})_L$  is revisited. We carefully reconsider the flat conserved currents, focusing upon the sign of the current improvement. In section 4 our previous analysis on the right description based on  $U(1)_R$  further proceeds. An infinite-dimensional extension of  $q$ -deformed Poincaré algebra is shown directly by constructing the conserved charges. The tower structure of the charges is generated in an *exotic* way. Then we show the gauge-equivalence of the monodromy matrices in the two descriptions and give a simple Riemann sphere interpretation of it. Section 5 is devoted to conclusion and discussion.

## 2 Setup

In this section we introduce the classical action of two-dimensional non-linear sigma models on three-dimensional Schrödinger spacetimes

### 2.1 Schrödinger spacetimes

Schrödinger spacetimes in three dimensions are known as null-like deformations of  $AdS_3$ . And the metric of the spacetimes is given by

$$ds^2 = L^2 \left[ -2e^{-2\rho} du dv + d\rho^2 - C e^{-4\rho} dv^2 \right]. \quad (2.1)$$

The deformation is measured by a real constant parameter  $C$ . When  $C = 0$ , the metric (2.1) describes the  $AdS_3$  space with the curvature radius  $L$  and the isometry is  $SO(2, 2) = SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ . When  $C \neq 0$ , the  $SL(2, \mathbb{R})_R$  symmetry is broken to  $U(1)_R$  while the  $SL(2, \mathbb{R})_L$  symmetry is preserved. Hence the symmetry that survives the deformation

is  $SL(2, \mathbb{R})_L \times U(1)_R$  in total. This symmetry yields two distinct descriptions to describe the classical dynamics as we will see later.

It is convenient to rewrite the metric (2.1) in terms of an  $SL(2, \mathbb{R})$  group element  $g$ ,

$$g = e^{2vT^+} e^{2\rho T^2} e^{2uT^-}, \quad T^\pm \equiv \frac{1}{\sqrt{2}} (T^0 \pm T^1). \quad (2.2)$$

Here the  $sl(2, \mathbb{R})$  generators  $T^a$  ( $a = 0, 1, 2$ ) satisfy

$$[T^a, T^b] = \varepsilon^{ab}{}_c T^c, \quad \text{Tr}(T^a T^b) = \frac{1}{2} \gamma^{ab}, \quad (2.3)$$

where  $\varepsilon^{ab}{}_c$  is the anti-symmetric tensor normalized as

$$\varepsilon^{01}{}_2 = +1, \quad \gamma^{ab} = \text{diag}(-1, +1, +1).$$

In the following discussion,  $\gamma^{ab}$  and its inverse are used to rising and lowering the  $sl(2, \mathbb{R})$  indices. It is useful to list the light-cone components of  $\varepsilon^{ab}{}_c$  and  $\gamma_{ab}$ :

$$\varepsilon^{-+}{}_2 = +1, \quad \gamma_{-+} = \gamma_{+-} = -1, \quad \gamma_{22} = +1.$$

For definiteness we take a representation of the  $sl(2, \mathbb{R})$  generators,

$$T^0 = \frac{i}{2} \sigma_2, \quad T^1 = \frac{1}{2} \sigma_1, \quad T^2 = \frac{1}{2} \sigma_3, \quad (2.4)$$

where  $\sigma_i$  ( $i = 1, 2, 3$ ) are the Pauli matrices and  $T^2$  is taken as the Cartan generator.

With the group element  $g$ , the left-invariant one-form  $J$  is defined as

$$J \equiv g^{-1} dg. \quad (2.5)$$

Now  $J$  is expanded in terms of the  $sl(2, \mathbb{R})$  generators like

$$\begin{aligned} J &= -T^0 J^0 + T^1 J^1 + T^2 J^2 \\ &= -T^+ J^- - T^- J^+ + T^2 J^2, \end{aligned}$$

where each of  $J^a$ 's is written in a simple form,

$$J^a = 2\text{Tr}(T^a J). \quad (2.6)$$

It is a turn to rewrite the metric (2.1) in terms of  $J$  like

$$ds^2 = \frac{L^2}{2} \left[ \text{Tr}(J^2) - 2C (\text{Tr}[T^- J])^2 \right]$$

$$= \frac{L^2}{4} [-2J^+ J^- + (J^2)^2 - C(J^-)^2] . \quad (2.7)$$

From the last expression (2.7), the Schrödinger spacetime can be regarded as a null-deformation of  $\text{AdS}_3$ .

Then the isometry  $SL(2, \mathbb{R})_{\text{L}} \times U(1)_{\text{R}}$  is represented by

$$g' = e^{\beta^a T^a} \cdot g \cdot e^{-\alpha T^-} ,$$

where  $\beta^a$  ( $a = 0, 1, 2$ ) and  $\alpha$  are real constant parameters. Its infinitesimal form is

$$\delta_L^a g = \epsilon_L^a T^a g , \quad \delta_R^- g = -\epsilon_R g T^- .$$

Here the summation is not taken for the repeated indices in the first equation.

## 2.2 Schrödinger sigma models

Our purpose is to investigate the classical integrable structure of two-dimensional non-linear sigma models on three-dimensional Schrödinger spacetime. For simplicity, we shall refer to these sigma models as *Schrödinger sigma models*.

With the metric (2.7), the classical action is given by

$$S = - \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \, \eta^{\mu\nu} [\text{Tr} (J_\mu J_\nu) - 2C \text{Tr} (T^- J_\mu) \text{Tr} (T^- J_\nu)] . \quad (2.8)$$

The base space is a two-dimensional Minkowski spacetime with the coordinates  $x^\mu = (t, x)$  and the metric  $\eta_{\mu\nu} = (-1, +1)$ . The rapidly dumping boundary condition is taken so that the group-valued field  $g(x)$  approaches a constant element  $g_\infty$  at spatial infinities:

$$g(x) \rightarrow g_\infty \quad (x \rightarrow \pm\infty) .$$

That is,  $J_\mu$  vanishes very rapidly at spatial infinities.

This setup is not appropriate in considering some applications to string theory. However, it is suitable to study infinite-dimensional symmetries generated by an infinite set of non-local charges in a well-defined way. The Virasoro constraints are also not taken into account.

Taking a variation of the action (2.8) leads to the equations of motion,

$$\partial^\mu J_\mu - 2C \text{Tr} (T^- \partial^\mu J_\mu) T^- - 2C \text{Tr} (T^- J_\mu) [J^\mu, T^-] = 0 . \quad (2.9)$$

By multiplying  $T^-$  and taking the trace, the conservation law of the  $U(1)_R$  current is obtained as

$$\partial^\mu J_\mu^- = 0. \quad (2.10)$$

With this conservation law (2.10), the equations of motion in (2.9) are simplified as

$$\partial^\mu J_\mu - 2C \text{Tr} (T^- J_\mu) [J^\mu, T^-] = 0. \quad (2.11)$$

When  $C = 0$ , the Schrödinger sigma models become  $SL(2, \mathbb{R})$  principal chiral models.

The Schrödinger sigma models are classically integrable [20]. The  $SL(2, \mathbb{R})_L$  and  $U(1)_R$  symmetries give rise to two descriptions to describe the classical dynamics. One is the left description based on  $SL(2, \mathbb{R})_L$ . The other is the right description based on  $U(1)_R$ . For each of them, Lax pairs and monodromy matrices are constructed. Then all Lax pairs lead to the identical equations of motion in (2.11).

The classical integrable structure is quite similar to the squashed  $S^3$  and warped  $\text{AdS}_3$  cases [23–26], but there is a significant difference. In the present case the latter description also belongs to the rational class, while in the squashed  $S^3$  and warped  $\text{AdS}_3$  cases it is trigonometric. Thus in this sense the classical integrable structure is not hybrid.

### 3 The left description

One way to describe the classical dynamics is the left description based on  $SL(2, \mathbb{R})_L$ . A pair of flat conserved currents is obtained by improving the  $SL(2, \mathbb{R})_L$  Noether current appropriately. Then the corresponding Lax pairs and monodromy matrices are constructed in the usual way. Both of them lead to the same equations of motion in (2.11) and two copies of the Yangians, and belong to the rational class.

This section is mainly a brief review of the previous work [20]. However, the sign of the improvement is carefully reconsidered here (while a specific sign has been considered in [20]) and it will be important in our later discussion.

#### 3.1 Flat conserved currents and Yangians

The key ingredient here is the  $SL(2, \mathbb{R})_L$  conserved current,

$$j_\mu^L = g J_\mu g^{-1} - 2C \text{Tr} (T^- J_\mu) g T^- g^{-1} + \epsilon_{\mu\nu} \partial^\nu f. \quad (3.1)$$

where  $\epsilon_{\mu\nu}$  is an anti-symmetric tensor normalized as  $\epsilon_{tx} = +1$  and  $f$  is an undetermined function. The first two terms are obtained by following the Noether's procedure and the last term is an improvement term.

An important point here is that the function  $f$  can be taken as

$$f = -AgT^-g^{-1},$$

so that the improved current satisfies the following equation,

$$\epsilon^{\mu\nu} (\partial_\mu j_\nu^L - j_\mu^L j_\nu^L) = (A^2 - C) \epsilon_{\mu\nu} \partial^\mu (gT^-g^{-1}) \partial^\nu (gT^-g^{-1}). \quad (3.2)$$

Thus, by choosing the appropriate values

$$A = \pm\sqrt{C}, \quad (3.3)$$

the improved current satisfies the flatness condition,

$$\epsilon^{\mu\nu} (\partial_\mu j_\nu^L - j_\mu^L j_\nu^L) = 0. \quad (3.4)$$

The explicit form of the resulting flat conserved currents are given by

$$j_\mu^{L\pm} = gJ_\mu g^{-1} - 2C \text{Tr} (T^- J_\mu) gT^-g^{-1} \mp \sqrt{C} \epsilon_{\mu\nu} \partial^\nu (gT^-g^{-1}). \quad (3.5)$$

The subscript of  $L$  (+ or  $-$ ) denotes the sign of the improvement term. Although only  $j_\mu^{L+}$  has been discussed in [20], both currents are important in the present case<sup>3</sup>.

With the improved currents, an infinite set of conserved non-local charges are recursively constructed by following [16]. The first three charges are given by

$$\begin{aligned} Q_{(0)}^L &= \int_{-\infty}^{\infty} dx \, j_t^{L\pm}(x), \\ Q_{(1)}^L &= \frac{1}{4} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \epsilon(x-y) [j_t^{L\pm}(x), j_t^{L\pm}(y)] - \int_{-\infty}^{\infty} dx \, j_x^{L\pm}(x), \\ Q_{(2)}^L &= \frac{1}{12} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \, \epsilon(x-y)\epsilon(x-z) [[j_t^{L\pm}(x), j_t^{L\pm}(y)], j_t^{L\pm}(z)] \\ &\quad - \frac{1}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \epsilon(x-y) [j_t^{L\pm}(x), j_x^{L\pm}(y)] + \int_{-\infty}^{\infty} dx \, j_t^{L\pm}(x), \\ &\vdots \end{aligned} \quad (3.6)$$

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<sup>3</sup>The current improvement leads to the flat conserved currents also in the squashed  $S^3$  case [27, 28]. This is the case even if the Wess-Zumino term has been added [29].

where  $\epsilon(x-y) \equiv \theta(x-y) - \theta(y-x)$  and  $\theta(x)$  is a step function. The subscript  $(n)$  denotes the degree of non-locality. Note that the sign of the improvement term is not relevant at the charge level because only  $C$  appears in the charges but  $\sqrt{C}$  does not. Thus the subscript  $\pm$  for the charges has been ignored above.

Then the current algebra is given in terms of the classical Poisson bracket  $\{ \cdot \}_P$ ,

$$\begin{aligned} \left\{ j_t^{L\pm,a}(x), j_t^{L\pm,b}(y) \right\}_P &= \epsilon^{ab} j_t^{L\pm,c}(x) \delta(x-y), \\ \left\{ j_t^{L\pm,a}(x), j_x^{L\pm,b}(y) \right\}_P &= \epsilon^{ab} j_x^{L\pm,c}(x) \delta(x-y) + \gamma^{ab} \partial_x \delta(x-y), \\ \left\{ j_x^{L\pm,a}(x), j_x^{L\pm,b}(y) \right\}_P &= 0. \end{aligned} \quad (3.7)$$

Here the components of  $j_\mu^{L\pm}$  are defined as

$$j_\mu^{L\pm,a} \equiv 2\text{Tr} \left( T^a j_\mu^{L\pm} \right).$$

Note that the current algebra (3.7) does not contain the deformation parameter  $C$ , in comparison to the squashed  $S^3$  and warped  $\text{AdS}_3$  cases. Thus in the present case it is straightforward to show that the charges (3.6) satisfy the defining relations of  $sl(2, \mathbb{R})$  Yangian in the sense of the first realization by Drinfeld [30].

### 3.2 Lax pairs and monodromy matrices

Let us next consider the Lax pairs. Since the flat conserved currents have already been obtained, it is easy to construct the Lax pairs,

$$L_\mu^{L\pm}(x; \lambda_{L\pm}) = \frac{1}{1 - \lambda_{L\pm}^2} \left( j_\mu^{L\pm}(x) - \lambda_{L\pm} \epsilon_{\mu\nu} j^{L\pm,\nu}(x) \right). \quad (3.8)$$

Here the spectral parameters  $\lambda_{L\pm}$  are constant complex numbers. Then the commutation relations

$$\left[ \partial_t - L_t^{L\pm}(x; \lambda_{L\pm}), \partial_x - L_x^{L\pm}(x; \lambda_{L\pm}) \right] = 0 \quad (3.9)$$

give rise to the flatness conditions and the conservation laws for the improved currents. We will refer to (3.8) as the left Lax pairs.

The flat conserved currents enable us to construct the monodromy matrices,

$$U^{L\pm}(\lambda_{L\pm}) = \text{P exp} \left[ \int_{-\infty}^{\infty} dx L_x^{L\pm}(x; \lambda_{L\pm}) \right], \quad (3.10)$$



where the symbol  $P$  denotes a path-ordering operation. It is straightforward to show that the monodromy matrices are conserved quantities,

$$\frac{d}{dt}U^{L\pm}(\lambda_{L\pm}) = 0.$$

Thus, by expanding (3.10) around a certain value of  $\lambda_{L\pm}$ , an infinite set of conserved charges are obtained. For instance, the  $sl(2, \mathbb{R})_L$  Yangian generators listed in (3.6) are reproduced by expanding (3.10) like

$$U^{L\pm}(\lambda_{L\pm}) = \exp \left[ \sum_{n=0}^{\infty} \lambda_{L\pm}^{-n-1} Q_{(n)}^L \right]$$

around  $\lambda_{L\pm} = \infty$ .

By evaluating the following Poisson brackets [31],

$$\begin{aligned} & \{L_x^{L\pm}(x; \lambda_{L\pm}) \otimes L_x^{L\pm}(y; \mu_{L\pm})\}_P \\ &= [r^{L\pm}(\lambda_{L\pm}, \mu_{L\pm}), L_x^{L\pm}(x; \lambda_{L\pm}) \otimes 1 + 1 \otimes L_x^{L\pm}(y; \mu_{L\pm})] \delta(x-y) \\ & \quad - [s^{L\pm}(\lambda_{L\pm}, \mu_{L\pm}), L_x^{L\pm}(x; \lambda_{L\pm}) \otimes 1 - 1 \otimes L_x^{L\pm}(y; \mu_{L\pm})] \delta(x-y) \\ & \quad - 2s^{L\pm}(\lambda_{L\pm}, \mu_{L\pm}) \partial_x \delta(x-y), \end{aligned}$$

we obtain the classical  $r/s$ -matrices of rational kind,

$$\begin{aligned} r^{L\pm}(\lambda_{L\pm}, \mu_{L\pm}) &= \frac{h(\lambda_{L\pm}) + h(\mu_{L\pm})}{2(\lambda_{L\pm} - \mu_{L\pm})} (-T^+ \otimes T^- - T^- \otimes T^+ + T^2 \otimes T^2), \\ s^{L\pm}(\lambda_{L\pm}, \mu_{L\pm}) &= \frac{h(\lambda_{L\pm}) - h(\mu_{L\pm})}{2(\lambda_{L\pm} - \mu_{L\pm})} (-T^+ \otimes T^- - T^- \otimes T^+ + T^2 \otimes T^2). \end{aligned}$$

Here we have introduced a scalar function,

$$h(\lambda) \equiv \frac{\lambda^2}{1 - \lambda^2} \tag{3.11}$$

The classical  $r/s$ -matrices satisfy the extended classical Yang-Baxter equation [20].

## 4 The right description

The other way to describe the classical dynamics is the right description based on  $U(1)_R$ . An infinite-dimensional symmetry should be related to this description. But it has not been clarified in the previous work [20], though the finite-dimensional part is found to be  $q$ -deformed Poincaré algebra.

Hence, after giving a short review on the previous result, we first consider its infinite-dimensional extension. By constructing the conserved charges explicitly, the *exotic* tower structure of the conserved charges will be revealed. Then we construct the associated Lax pairs and monodromy matrices, which are gauge-equivalent to the ones in the left description. We also give a simple Riemann sphere description of this equivalence.

## 4.1 $q$ -deformed Poincaré algebra

The conserved current related to the  $U(1)_R$  symmetry is given by

$$j_\mu^{R,-} = -J_\mu^- . \quad (4.1)$$

For the broken components of  $SL(2, \mathbb{R})_R$ , the following non-local currents

$$\begin{aligned} j_\mu^{R,2} &= -e^{\sqrt{C}\chi} \left( J_\mu^2 + \sqrt{C} \epsilon_{\mu\nu} J^{-,\nu} \right) , \\ j_\mu^{R,+} &= -e^{\sqrt{C}\chi} \left( J_\mu^+ + \sqrt{C} \epsilon_{\mu\nu} J^{2,\nu} + C J_\mu^- \right) \end{aligned} \quad (4.2)$$

are conserved [20]. Here we have introduced a non-local field  $\chi(x)$  defined as

$$\chi(x) \equiv -\frac{1}{2} \int_{-\infty}^{\infty} dy \, \epsilon(x-y) j_t^{R,-}(y) . \quad (4.3)$$

Thus the conserved charges are

$$Q^{R,-} = \int_{-\infty}^{\infty} dx \, j_t^{R,-}(x) , \quad Q^{R,2} = \int_{-\infty}^{\infty} dx \, j_t^{R,2}(x) , \quad Q^{R,+} = \int_{-\infty}^{\infty} dx \, j_t^{R,+}(x) . \quad (4.4)$$

Note that the last two charges are non-local due to the presence of (4.3).

Then the Poisson brackets of the charges are computed as [20]

$$\begin{aligned} \{Q^{R,+}, Q^{R,-}\}_P &= -Q^{R,2} , \\ \{Q^{R,+}, Q^{R,2}\}_P &= -Q^{R,+} \cosh \left( \frac{\sqrt{C}}{2} Q^{R,-} \right) , \\ \{Q^{R,-}, Q^{R,2}\}_P &= \frac{2}{\sqrt{C}} \sinh \left( \frac{\sqrt{C}}{2} Q^{R,-} \right) . \end{aligned} \quad (4.5)$$

This algebra (4.5) is not the standard  $q$ -deformation of  $sl(2)$  [30,32], because this is not for a deformation for the Cartan direction. After taking an appropriate rescaling of the charges,

the algebra (4.5) is isomorphic to a two-dimensional  $q$ -deformed Poincaré algebra [33, 34], where the deformation parameter  $q$  is defined as

$$q \equiv e^{\frac{\sqrt{C}}{2}}.$$

Eventually the algebra (4.5) is also known as a non-standard  $q$ -deformation of  $sl(2)$ .

Note that the non-local currents in (4.2) are related to  $j_\mu^{L+}$  through a non-local map<sup>4</sup>

$$\begin{aligned} j_\mu^{R,2} &= -2e^{\sqrt{C}\chi} \text{Tr} \left( T^2 g^{-1} j_\mu^{L+} g \right), \\ j_\mu^{R,+} &= -2e^{\sqrt{C}\chi} \text{Tr} \left( T^+ g^{-1} j_\mu^{L+} g \right). \end{aligned} \quad (4.6)$$

One can directly show that the currents in (4.6) are conserved. This implies that the left-right duality in  $SL(2)$  principal chiral models is realized in a non-local way even after the deformation has been performed.

## 4.2 Exotic infinite-dimensional extension

Our main purpose is to explore an infinite-dimensional extension of the algebra (4.5). That is, we would like to consider its affine extension.

To find out the candidate of affine generators, let us try a non-local map to  $j_\mu^{L-}$  like

$$\begin{aligned} \tilde{j}_\mu^{R,2} &= -2e^{-\sqrt{C}\chi} \text{Tr} \left( T^2 g^{-1} j_\mu^{L-} g \right), \\ \tilde{j}_\mu^{R,+} &= -2e^{-\sqrt{C}\chi} \text{Tr} \left( T^+ g^{-1} j_\mu^{L-} g \right). \end{aligned} \quad (4.7)$$

The resulting non-local currents are given by

$$\begin{aligned} \tilde{j}_\mu^{R,2} &= -e^{-\sqrt{C}\chi} \left( J_\mu^2 - \sqrt{C} \epsilon_{\mu\nu} J^{-,\nu} \right), \\ \tilde{j}_\mu^{R,+} &= -e^{-\sqrt{C}\chi} \left( J_\mu^+ - \sqrt{C} \epsilon_{\mu\nu} J^{2,\nu} + C J_\mu^- \right), \end{aligned} \quad (4.8)$$

and it is an easy task to show that these are conserved. The corresponding conserved charges are defined as

$$\tilde{Q}^{R,2} = \int_{-\infty}^{\infty} dx \tilde{j}_t^{R,2}(x) \equiv Q_{(1)}^{R,2}, \quad \tilde{Q}^{R,+} = \int_{-\infty}^{\infty} dx \tilde{j}_t^{R,+}(x) \equiv Q_{(2)}^{R,+}. \quad (4.9)$$

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<sup>4</sup> The non-local map has been found originally in the case of squashed sigma models [23].

The meaning of the subscripts (1) and (2) will be clarified later as the level of the resulting algebra. For convenience we rename the generators of  $q$ -deformed Poincaré algebra as

$$Q_{(0)}^{R,-} \equiv Q^{R,-}, \quad Q_{(0)}^{R,2} \equiv Q^{R,2}, \quad Q_{(0)}^{R,+} \equiv Q^{R,+}. \quad (4.10)$$

The tower structure generated by (4.9) and (4.10) can be deduced by evaluating the Poisson brackets without the knowledge on the mathematical formulation of the affine extension of  $q$ -deformed Poincaré algebra. This is a great advantage of the sigma model calculation.

We shall show some examples of the Poisson brackets below. Those are enough to deduce the tower structure of the conserved charges.

The first is the Poisson bracket of  $Q_{(0)}^{R,+}$  and  $Q_{(1)}^{R,2}$ ,

$$\left\{ Q_{(0)}^{R,+}, Q_{(1)}^{R,2} \right\}_P = Q_{(1)}^{R,+} + \cosh \left( \frac{\sqrt{C}}{2} Q_{(0)}^{R,-} \right) Q_{(0)}^{R,+},$$

where a new conserved charge is defined as

$$Q_{(1)}^{R,+} \equiv \frac{\sqrt{C}}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \epsilon(x-y) j_t^{R,2}(x) \tilde{j}_t^{R,2}(y) + 2 \int_{-\infty}^{\infty} dx \, J_t^+(x). \quad (4.11)$$

Then let us compute the following bracket,

$$\left\{ Q_{(0)}^{R,+}, Q_{(1)}^{R,+} \right\}_P = Q_{(1)}^{R,++} - \frac{\sqrt{C}}{2} \sinh \left( \frac{\sqrt{C}}{2} Q_{(0)}^{R,-} \right) Q_{(0)}^{R,2} Q_{(0)}^{R,+}, \quad (4.12)$$

where a new conserved charge  $Q_{(1)}^{R,++}$  is given by

$$\begin{aligned} Q_{(1)}^{R,++} \equiv & -\frac{C}{4} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \, \epsilon(x-y) j_t^{R,2}(x) j_t^{R,2}(y) \tilde{j}_t^{R,2}(z) \\ & + \sqrt{C} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \epsilon(x-y) j_t^{R,2}(x) J_t^+(y) - 2\sqrt{C} \int_{-\infty}^{\infty} dx \, j_x^{R,+}(x). \end{aligned} \quad (4.13)$$

Note that  $Q_{(1)}^{R,++}$  can also be obtained by evaluating the following bracket,

$$\begin{aligned} \left\{ Q_{(0)}^{R,+}, \left\{ Q_{(0)}^{R,+}, Q_{(1)}^{R,2} \right\}_P \right\}_P = & Q_{(1)}^{R,++} - \sqrt{C} \sinh \left( \frac{\sqrt{C}}{2} Q_{(0)}^{R,-} \right) Q_{(0)}^{R,2} Q_{(0)}^{R,+} \\ & + \frac{\sqrt{C}}{2} \sinh \left( \frac{\sqrt{C}}{2} Q_{(0)}^{R,-} \right) \tilde{Q}^{R,2} Q_{(0)}^{R,+} + \frac{C}{4} (Q_{(0)}^{R,2})^2 \tilde{Q}^{R,2}. \end{aligned}$$

By multiplying  $Q_{(0)}^{R,+}$  to  $Q_{(1)}^{R,++}$ , we obtain that

$$\begin{aligned} \left\{ Q_{(0)}^{R,+}, Q_{(1)}^{R,++} \right\}_P &= \frac{\sqrt{C}}{2} \sinh \left( \frac{\sqrt{C}}{2} Q_{(0)}^{R,-} \right) Q_{(0)}^{R,+} Q_{(1)}^{R,+} \\ &\quad + \frac{C}{4} \cosh \left( \frac{\sqrt{C}}{2} Q_{(0)}^{R,-} \right) Q_{(0)}^{R,2} Q_{(1)}^{R,2} Q_{(0)}^{R,+}. \end{aligned}$$

Hence any new conserved charges have not been generated. This result can be recast into the Serre relation,

$$\begin{aligned} &\left\{ Q_{(0)}^{R,+}, \left\{ Q_{(0)}^{R,+}, \left\{ Q_{(0)}^{R,+}, Q_{(1)}^{R,2} \right\}_P \right\}_P \right\}_P \\ &= \sqrt{C} \sinh \left( \frac{\sqrt{C}}{2} Q_{(0)}^{R,-} \right) Q_{(0)}^{R,+} \left\{ Q_{(0)}^{R,+}, Q_{(1)}^{R,2} \right\}_P + \frac{C}{4} \left( Q_{(0)}^{R,2} \right)^2 \left\{ Q_{(0)}^{R,+}, Q_{(1)}^{R,2} \right\}_P \\ &\quad + \frac{C}{2} \cosh \left( \frac{\sqrt{C}}{2} Q_{(0)}^{R,-} \right) \left( Q_{(0)}^{R,2} \right)^2 Q_{(0)}^{R,+} - \frac{C}{2} \cosh \left( \frac{\sqrt{C}}{2} Q_{(0)}^{R,-} \right) Q_{(0)}^{R,2} Q_{(1)}^{R,2} Q_{(0)}^{R,+} \\ &\quad + \frac{\sqrt{C}}{2} \sinh \left( \frac{\sqrt{C}}{2} Q_{(0)}^{R,-} \right) \cosh \left( \frac{\sqrt{C}}{2} Q_{(0)}^{R,-} \right) \left( Q_{(0)}^{R,+} \right)^2. \end{aligned} \tag{4.14}$$

Thus evaluating possible Poisson brackets enables us to deduce the tower structure of the conserved charges in Figure 1, up to lower level conserved charges. The level is defined as the necessary number of  $\tilde{Q}^{R,2}$  ( $= Q_{(1)}^{R,2}$ ) to construct the charges at the level from the charges of  $q$ -deformed Poincaré algebra (This is the definition of the level 0 charges). The number of  $+$  is basically measured by the number of charges with the  $+$  group index but the  $+$  index should be formally assigned to  $\sqrt{C}$  so as to define the number of  $+$  properly.

Figure 1 seems a tilted Yangian algebra, but it contains a constant deformation parameter and the level 0 part is given by a  $q$ -deformed Poincaré algebra rather than  $SL(2, \mathbb{R})_R$ , so the resulting tower structure seems to be different from the well-known Yangian (though there might be an isomorphism). In this sense this algebra should be called *exotic*.

Note that  $\tilde{Q}^{R,+}$  ( $= Q_{(2)}^{R,+}$ ) looks like an affine generator at a glance from the definition in analogy with the squashed  $S^3$  case [24]. However, it is not necessary to include  $\tilde{Q}^{R,+}$  into the defining relations so as to generate the whole tower of conserved charges. Eventually it is enough to take only  $Q_{(1)}^{R,2}$ . In fact,  $\tilde{Q}^{R,+}$  ( $= Q_{(2)}^{R,+}$ ) is expressed in terms of lower conserved charges like

$$\left\{ Q_{(1)}^{R,2}, Q_{(1)}^{R,+} \right\}_P = -\tilde{Q}^{R,+} - \frac{\sqrt{C}}{2} \sinh \left( \frac{\sqrt{C}}{2} Q_{(0)}^{R,-} \right) \left( Q_{(1)}^{R,2} \right)^2 \tag{4.15}$$

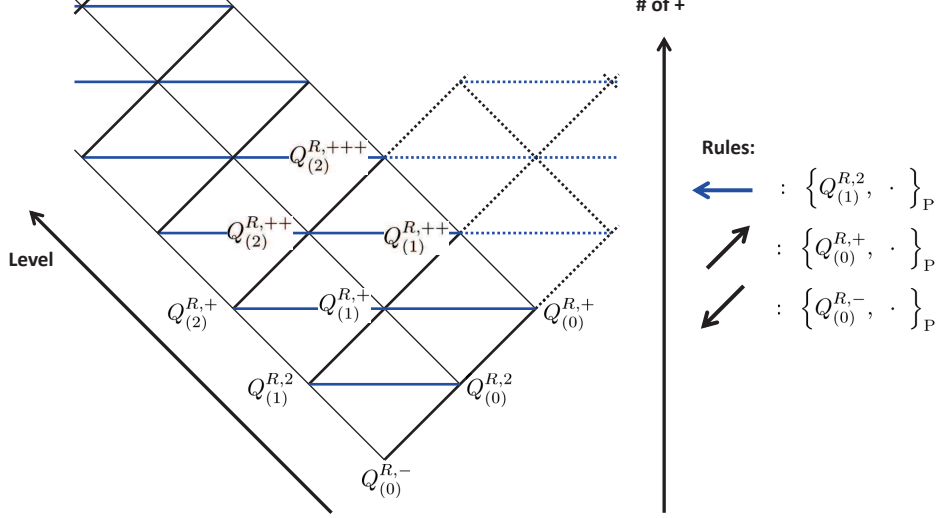


Figure 1: The tower structure of conserved charges.

$$+ \frac{\sqrt{C}}{4} \sinh \left( \frac{\sqrt{C}}{2} Q_{(0)}^{R,-} \right) Q_{(0)}^{R,2} Q_{(1)}^{R,2}.$$

and can be regarded as one of the level 2 conserved charges. This is the reason that we have assigned the subscript (2) as  $\tilde{Q}^{R,+} = Q_{(2)}^{R,+}$ . An interpretation why the structure of this kind happens will be explained from the viewpoint of Riemann spheres later.

There is another way to represent the tower of conserved charges. It is worth noting that the charges  $\tilde{Q}^{R,\pm}$  and  $\tilde{Q}^{R,2}$  also generate  $q$ -deformed Poincaré algebra. Hence, instead of  $Q^{R,\pm}$  and  $Q^{R,2}$ , one can use  $\tilde{Q}^{R,\pm}$  and  $\tilde{Q}^{R,2}$  as the level 0 charges. We refer to them as  $\tilde{Q}_{(0)}^{R,-}$ ,  $\tilde{Q}_{(0)}^{R,2}$ , and  $\tilde{Q}_{(0)}^{R,+}$ . Then  $Q^{R,2}$  plays a role of an affine generator in turn and should be called  $\tilde{Q}_{(1)}^{R,2}$ . The tower structure is depicted in Figure 2.

It would probably not be difficult to notice that there is a certain relation between the two representations. In fact, these are exchanged through the sign flip of  $\sqrt{C}$  like

$$\sqrt{C} \longleftrightarrow -\sqrt{C}. \quad (4.16)$$

More concretely speaking, the flipping (4.16) exchanges the charges as

$$Q_{(n)}^{R,*} \longleftrightarrow \tilde{Q}_{(n)}^{R,*} \quad (* = \pm, 2).$$

Thus the one representation is dual to the other one through (4.16).

Naively looking at the tower structure, one might think of that the two towers should be merged into a single V shape tower. However, it is not the case. To understand it

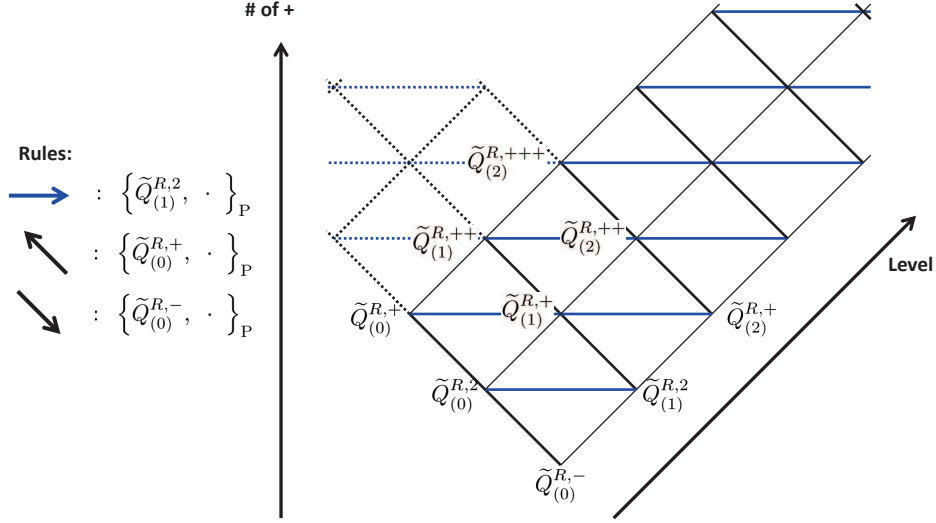


Figure 2: Another expression of the whole tower of conserved charges.

accurately, it is of great value to note on the following Poisson brackets, as an example,

$$\left\{ \tilde{Q}_{(1)}^{R,2}, \tilde{Q}_{(0)}^{R,+} \right\}_P = -\tilde{Q}_{(1)}^{R,+} - \cosh \left( \frac{\sqrt{C}}{2} \tilde{Q}_{(0)}^{R,-} \right) \tilde{Q}_{(0)}^{R,+}, \quad (4.17)$$

$$\left\{ Q_{(0)}^{R,2}, Q_{(2)}^{R,+} \right\}_P = -Q_{(1)}^{R,+} - \cosh \left( \frac{\sqrt{C}}{2} Q_{(0)}^{R,-} \right) Q_{(2)}^{R,+}. \quad (4.18)$$

The two equations are precisely identical but only the level assignment is different. The difference of the level assignment is very important because the interpretation of the right-hand sides of (4.17) and (4.18) depends on it.

The first relation (4.17) implies that  $\tilde{Q}_{(1)}^{R,+}$  is obtained when  $\tilde{Q}_{(1)}^{R,2}$  acts on  $\tilde{Q}_{(0)}^{R,+}$ . The right-hand side of (4.17) is interpreted that acting  $\tilde{Q}_{(1)}^{R,2}$  gives rise to the movement to right by one, because the lower conserved charges are discarded. This is depicted in Figure 2.

On the other hand, the second relation (4.18) says that acting  $Q_{(0)}^{R,2}$  on  $Q_{(2)}^{R,+}$  does not generate any new charge. At a glance, it seems that the right-hand side of (4.18) contains  $Q_{(1)}^{R,+}$  sitting at the right of  $Q_{(2)}^{R,+}$  and hence this operation also generates the movement to the right. However,  $Q_{(1)}^{R,+}$  is lower than  $Q_{(2)}^{R,+}$  in the level assignment in Figure 1. Thus  $Q_{(2)}^{R,+}$  is picked up and after all  $Q_{(2)}^{R,+}$  keeps staying under the action of  $Q_{(0)}^{R,2}$ . In fact, any charges remain the same position under the action of  $Q_{(0)}^{R,2}$  and the tower structure in Figure 1 is determined.

This is the trick to determine the tower structure that is semi-infinite in one direction

like Yangians rather than quantum affine algebras. The semi-infinite behavior is typical for the rational integrable models and it should be related to the fact that the  $r/s$ -matrices are of rational kind in the right description.

Finally we comment on the Yangian limit. When taking the  $C \rightarrow 0$  limit, the  $SL(2, \mathbb{R})_R$  Yangian should be reproduced from the exotic symmetry discussed here. This limit is not obvious in the representations given in Figures 1 and 2. The level 0 charges in the  $SL(2, \mathbb{R})_R$  Yangian are obtained from the charges with  $2, \pm$ . Then the level 1 charges are obtained from the charges with more  $+$ . For example, the level 1 charge with  $+$  is obtained from  $Q_{(1)}^{R,++}$  in (4.13) by multiplying  $1/\sqrt{C}$  before taking the  $C \rightarrow 0$  limit. In fact, there is a better basis of the charges so that the Yangian limit is manifest. For this basis, it will be explained in [35].

### 4.3 The right Lax pairs

It is a turn to consider Lax pairs in the right description. We refer to the Lax pairs as the right Lax pairs. Indeed, there are some ways to derive the right Lax pairs. Here we shall take a scaling limit of space-like or time-like warped  $AdS_3$  spaces. Here we follow a different derivation from the one discussed in [20] because the argument on the left-right duality becomes much clearer.

The metrics of space-like and time-like warped  $AdS_3$  are given by, respectively [36],

$$ds^2 = \frac{L^2}{2} \left[ \text{Tr} (J^2) - 2\tilde{C} (\text{Tr} [T^1 J])^2 \right] \quad (\text{space-like}), \quad (4.19)$$

$$ds^2 = \frac{L^2}{2} \left[ \text{Tr} (J^2) - 2\tilde{C} (\text{Tr} [T^0 J])^2 \right] \quad (\text{time-like}). \quad (4.20)$$

The constant parameter  $\tilde{C}$  is a deformation parameter. When  $\tilde{C} = 0$ , the metrics (4.19) and (4.20) degenerate to the one of  $AdS_3$  with radius  $L$ .

Let us start from a scaling limit of the space-like warped  $AdS_3$  to the Schrödinger spacetime (The process is the same for the time-like). First, the metric (4.19) is rewritten in terms of  $T^\pm$  as

$$ds^2 = \frac{L^2}{2} \left[ \text{Tr} (J^2) - \tilde{C} (\text{Tr} [T^- J])^2 + 2\tilde{C} \text{Tr} [T^- J] \text{Tr} [T^+ J] - \tilde{C} (\text{Tr} [T^+ J])^2 \right].$$

Then  $T^\pm$  is rescaled like

$$T^- \rightarrow \sqrt{\frac{2C}{\tilde{C}}} T^-, \quad T^+ \rightarrow \sqrt{\frac{\tilde{C}}{2C}} T^+. \quad (4.21)$$



The resulting metric is given by

$$ds^2 = \frac{L^2}{2} \left[ \text{Tr} (J^2) - 2C (\text{Tr} [T^- J])^2 + 2\tilde{C} \text{Tr} [T^- J] \text{Tr} [T^+ J] - \frac{\tilde{C}^2}{2C} (\text{Tr} [T^+ J])^2 \right].$$

Finally, by taking the following limit

$$\tilde{C} \rightarrow 0 \quad \text{with } C \text{ fixed},$$

the metric of the Schrödinger spacetime (2.7) is reproduced.

Our strategy here is to apply the same scaling limit to the Lax pairs in warped  $\text{AdS}_3$  sigma models so as to obtain the right Lax pairs of the Schrödinger sigma models. The classical action of two-dimensional non-linear sigma models defined on space-like warped  $\text{AdS}_3$  is given by

$$S = - \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \eta^{\mu\nu} \left[ \text{Tr} (J_\mu J_\nu) - 2\tilde{C} \text{Tr} (T^1 J_\mu) \text{Tr} (T^1 J_\nu) \right]. \quad (4.22)$$

The right Lax pair is basically given in [37, 38] and its expression is

$$L_\pm^R(x; \lambda_R) = -\frac{\sinh \alpha}{\sinh (\alpha \pm \lambda_R)} \left[ -T^0 J_\pm^0 + T^2 J_\pm^2 + \frac{\cosh (\alpha \pm \lambda_R)}{\cosh \alpha} T^1 J_\pm^1 \right], \quad (4.23)$$

$$J_\pm = J_t \pm J_x, \quad \sqrt{\tilde{C}} = \tanh \alpha.$$

Then by using the isomorphism of the  $sl(2)$  algebra [25]

$$(T^0 \pm T^2) \rightarrow e^{\mp \lambda_R} (T^0 \pm T^2), \quad (4.24)$$

the Lax pair (4.23) is rewritten as

$$L_\pm^{R+}(x; \lambda_{R+}) = -\frac{\sinh \alpha}{\sinh (\alpha \pm \lambda_{R+})} \left[ -\frac{1}{2} e^{+\lambda_{R+}} (T^0 - T^2) (J_\pm^0 + J_\pm^2) \right. \\ \left. -\frac{1}{2} e^{-\lambda_{R+}} (T^0 + T^2) (J_\pm^0 - J_\pm^2) + \frac{\cosh (\alpha \pm \lambda_{R+})}{\cosh \alpha} T^1 J_\pm^1 \right], \quad (4.25)$$

where  $\lambda_R$  has been renamed  $\lambda_{R+}$  because the fundamental domain of  $\lambda_{R+}$  is now just half of the one of  $\lambda_R$ , namely the periodicity of  $\lambda_{R+}$  is  $\pi$  while that of  $\lambda_R$  is  $2\pi$ . According to this division, the number of poles contained in the Lax pair is also reduced from four to two. Thus the isomorphism (4.24) leads to the decomposition of the trigonometric Lax pair into a pair of the rational Lax pairs [25]. The Lax pair (4.25) is one of the rational Lax pairs.

The other rational Lax pair can be obtained through the  $sl(2)$  isomorphism

$$(T^0 \pm T^2) \rightarrow e^{\pm \lambda_R} (T^0 \pm T^2) . \quad (4.26)$$

The resulting rational Lax pair is given by

$$\begin{aligned} L_{\pm}^{R-}(x; \lambda_{R-}) = & -\frac{\sinh \alpha}{\sinh(\alpha \pm \lambda_{R-})} \left[ -\frac{1}{2} e^{-\lambda_{R-}} (T^0 - T^2) (J_{\pm}^0 + J_{\pm}^2) \right. \\ & \left. -\frac{1}{2} e^{+\lambda_{R-}} (T^0 + T^2) (J_{\pm}^0 - J_{\pm}^2) + \frac{\cosh(\alpha \pm \lambda_{R-})}{\cosh \alpha} T^1 J_{\pm}^1 \right] , \end{aligned} \quad (4.27)$$

where  $\lambda_R$  has been renamed  $\lambda_{R-}$  on the same score.

The scaling limits for (4.25) and (4.27) give rise to the following right Lax pairs,

$$\begin{aligned} L_{\mu}^{R\pm}(\lambda_{R\pm}) = & \frac{1}{1 - \lambda_{R\pm}^2} \left[ -T^+ (-J_{\mu}^- + \lambda_{R\pm} \epsilon_{\mu\nu} J^{-,\nu}) \right. \\ & + T^2 \left( -J_{\mu}^2 + \lambda_{R\pm} \epsilon_{\mu\nu} J^{2,\nu} \pm \sqrt{C} \lambda_{R\pm} J_{\mu}^- \mp \sqrt{C} \lambda_{R\pm}^2 \epsilon_{\mu\nu} J^{-,\nu} \right) \\ & - T^- \left( -J_{\mu}^+ + \lambda_{R\pm} \epsilon_{\mu\nu} J^{+,\nu} \pm \sqrt{C} \lambda_{R\pm} J_{\mu}^2 \mp \sqrt{C} \lambda_{R\pm}^2 \epsilon_{\mu\nu} J^{2,\nu} \right. \\ & \left. \left. + C \lambda_{R\pm} \epsilon_{\mu\nu} J^{-,\nu} - C \lambda_{R\pm}^2 J_{\mu}^- \right) \right] , \end{aligned} \quad (4.28)$$

where we have rescaled  $\lambda_{R\pm}$  as  $\lambda_{R\pm} \rightarrow \alpha \lambda_{R\pm}$  before taking the scaling limit. The Lax pairs (4.28) are also obtained by performing the following isomorphism

$$\begin{aligned} T^- & \rightarrow T^- , \\ T^2 & \rightarrow T^2 \mp \sqrt{C} \lambda_{R\pm} T^- , \\ T^+ & \rightarrow T^+ \mp \sqrt{C} \lambda_{R\pm} T^2 + \frac{C}{2} \lambda_{R\pm}^2 T^- , \end{aligned} \quad (4.29)$$

to the right Lax pair in [20].

It is easy to check that the equations of motion (2.11) are reproduced from the commutation relations

$$\left[ \partial_t - L_t^{R\pm}(x; \lambda_{R\pm}), \partial_x - L_x^{R\pm}(\lambda_{R\pm}) \right] = 0 . \quad (4.30)$$

Then the monodromy matrices are given by

$$U^{R\pm}(\lambda_{R\pm}) = \text{P exp} \left[ \int_{-\infty}^{\infty} dx L_x^{R\pm}(x; \lambda_{R\pm}) \right] . \quad (4.31)$$

These are conserved quantities and by expanding them with respect to  $\lambda_{R,\pm}$  an infinite set of conserved charges are obtained. By following the procedure in [31], the right  $r/s$ -matrices are given by

$$\begin{aligned}
r^{R,\pm}(\lambda_{R,\pm}, \mu_{R,\pm}) &= (h(\lambda_{R,\pm}) + h(\mu_{R,\pm})) \\
&\times \left[ \frac{1}{2(\lambda_{R,\pm} - \mu_{R,\pm})} (-T^+ \otimes T^- - T^- \otimes T^+ + T^2 \otimes T^2) \right. \\
&\quad \left. \pm \frac{\sqrt{C}}{4} (T^- \otimes T^2 - T^2 \otimes T^-) \right], \\
s^{R,\pm}(\lambda_{R,\pm}, \mu_{R,\pm}) &= (h(\lambda_{R,\pm}) - h(\mu_{R,\pm})) \\
&\times \left[ \frac{1}{2(\lambda_{R,\pm} - \mu_{R,\pm})} (-T^+ \otimes T^- - T^- \otimes T^+ + T^2 \otimes T^2) \right. \\
&\quad \left. \pm \frac{\sqrt{C}}{4} (T^- \otimes T^2 - T^2 \otimes T^-) \right],
\end{aligned}$$

with  $h(\lambda)$  defined in (3.11). These are of rational kind and satisfy the extended classical Yang-Baxter equation.

Thus there are the two rational Lax pairs in the right description. Recall that there are two rational Lax pairs also in the left description. In fact, there is a relation between the right and left descriptions as we will see in the next subsection.

#### 4.4 The gauge equivalence and Riemann sphere

So far, we have seen that there are the two descriptions 1) the left description and 2) the right description. Let us show that the two descriptions are related through a gauge transformation and eventually equivalent.

Before arguing the gauge transformation, it is necessary to fix the relations between the spectral parameters in the two descriptions. These are obtained by taking a scaling limit of the spectral parameter relations, for example, in the case of space-like warped  $\text{AdS}_3$  sigma models,

$$\lambda_{L\pm} = \frac{\tanh \alpha}{\tanh \lambda_{R\pm}} \quad (\text{warped AdS}_3).$$

The resulting relations after the scaling limit are simply given by

$$\lambda_{L\pm} = \frac{1}{\lambda_{R\pm}} \quad (\text{Schrödinger}). \quad (4.32)$$

This is the same as in the case of principal chiral models, up to the subscript  $\pm$ .

Now it is a turn to show the gauge equivalence. With the relations in (4.32), the right Lax pairs are rewritten as

$$\begin{aligned} L_\mu^{R\pm}(\lambda_{R\pm}) = & -\frac{1}{1-\lambda_{L\pm}^2} \left[ -T^+ \left( -\lambda_{L\pm}^2 J_\mu^- + \lambda_{L\pm} \epsilon_{\mu\nu} J^{-,\nu} \right) \right. \\ & + T^2 \left( -\lambda_{L\pm}^2 J_\mu^2 + \lambda_{L\pm} \epsilon_{\mu\nu} J^{2,\nu} \pm \sqrt{C} \lambda_{L\pm} J_\mu^- \mp \sqrt{C} \epsilon_{\mu\nu} J^{-,\nu} \right) \\ & - T^- \left( -\lambda_{L\pm}^2 J_\mu^+ + \lambda_{L\pm} \epsilon_{\mu\nu} J^{+,\nu} \pm \sqrt{C} \lambda_{L\pm} J_\mu^2 \mp \sqrt{C} \epsilon_{\mu\nu} J^{2,\nu} \right. \\ & \left. \left. + C \lambda_{L\pm} \epsilon_{\mu\nu} J^{-,\nu} - C J_\mu^- \right) \right]. \end{aligned}$$

Then let us perform the gauge transformation like

$$\begin{aligned} \left[ L_\mu^{R\pm}(\lambda_{R\pm}) \right]^g & \equiv \partial_\mu g \cdot g^{-1} + g L_\mu^{R\pm}(\lambda_{R\pm}) g^{-1} \\ & = \frac{1}{1-\lambda_{L\pm}^2} g \left\{ -T^+ \left[ J_\mu^- - \lambda_{L\pm} \epsilon_{\mu\nu} J^{-,\nu} \right] \right. \\ & \quad + T^2 \left[ \left( J_\mu^2 \pm \sqrt{C} \epsilon_{\mu\nu} J^{-,\nu} \right) - \lambda_{L\pm} \epsilon_{\mu\nu} \left( J^{2,\nu} \pm \sqrt{C} \epsilon^{\nu\rho} J_\rho^- \right) \right] \\ & \quad \left. - T^- \left[ \left( J_\mu^+ \pm \sqrt{C} \epsilon_{\mu\nu} J^{2,\nu} + C J_\mu^- \right) - \lambda_{L\pm} \epsilon_{\mu\nu} \left( J^{+,\nu} \pm \sqrt{C} \epsilon^{\nu\rho} J_\rho^2 + C J^{-,\nu} \right) \right] \right\} g^{-1} \\ & = L_\mu^{L\pm}(\lambda_{L\pm}). \end{aligned}$$

Thus the Lax pairs in the left and right descriptions are gauge-equivalent,

$$\left[ L_\mu^{R\pm}(\lambda_{R\pm}) \right]^g = L_\mu^{L\pm}(\lambda_{L\pm}). \quad (4.33)$$

At the level of monodromy matrix, (4.33) is recast into

$$U^{R\pm}(\lambda_\pm) = g_\infty^{-1} \cdot U^{L\pm}(\lambda_\pm) \cdot g_\infty. \quad (4.34)$$

Thus the right description is gauge-equivalent globally to the left one.

We comment on the Riemann sphere picture of this equivalence as depicted in Figure 3. The pole structure of the Lax pairs is figured out from the expressions. For each of them, the space of spectral parameter is represented by a Riemann sphere with two punctures. The spectral parameter relations (4.32) lead to the equivalence of the Riemann spheres of  $\lambda_{R+}$  and  $\lambda_{L+}$ . Similar equivalence holds for  $\lambda_{R-}$  and  $\lambda_{L-}$ .

Now that we have two independent Riemann spheres with two punctures for the subscripts  $+$  and  $-$ . Note that the two spheres are related one another through the sign

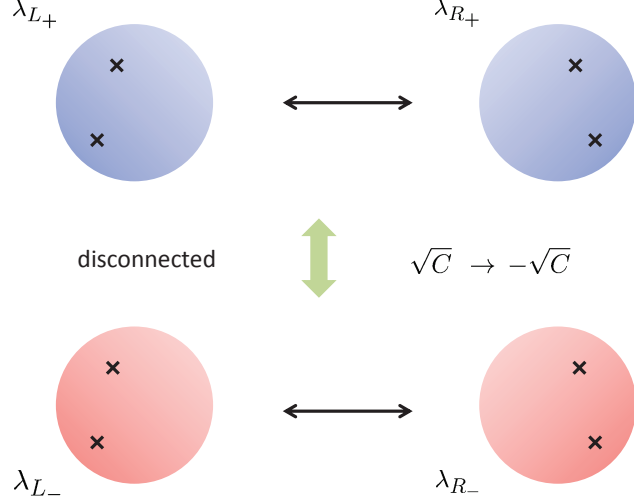


Figure 3: The spaces of spectral parameters are described as Riemann spheres with two punctures. The blue spheres are gauge-equivalent to the red ones.

flipping of  $\sqrt{C}$ . Then the one representation of the exotic symmetry comes from the Riemann sphere for  $+$  and the other one comes from that for  $-$ .

However, this sign flipping is equivalent to the following  $sl(2)$  isomorphism and the field redefinition,

$$T^{\pm} \rightarrow -T^{\pm}, \quad J^{\pm} \rightarrow -J^{\pm}. \quad (4.35)$$

This means that the sign flipping of  $\sqrt{C}$  is written as a gauge transformation. Thus the two Riemann spheres with  $+$  and  $-$  are also gauge-equivalent. For this reason, one of the candidate of affine generators is realized as one of the second level conserved charges. That is, the level 0 charges with  $+$  and 2 on the one side are realized as level 2 charges on the other side through the gauge transformation. Thus the two representations of the exotic symmetry are also gauge-equivalent. In other words, the level assignment is nothing but a gauge degree of freedom.

The resulting geometry in Figure 3 can also be interpreted as a scaling limit of the Riemann spheres in the case of warped  $AdS_3$  sigma models (See [25]). Because the scaling limit contains the  $\tilde{C} \rightarrow 0$  limit, the cut shrinks to a point. After that, the geometry is given by Figure 3.

Finally we shall summarize the conserved charges of infinite-dimensional symmetries and the expansion points of the monodromy matrices in Table 1. It is easy to understand

the expansion points for the  $sl(2, \mathbb{R})_L$  Yangian with the relations between the spectral parameters (4.32). The conserved charges for the affine  $q$ -deformed Poincaré algebra can also be obtained by expanding the right monodromy matrices  $U^{R\pm}(\lambda_{R\pm})$  around  $\lambda_{R\pm} = \infty$  [35]. According to this, the left expansion point is automatically determined through (4.32). Actually, there is a slight difference between the charges obtained in the subsection 4.2 and the ones obtained from the monodromy matrices. The two sets of the charges are homomorphic but not always isomorphic depending on the sign of  $C$ . The details will be reported in [35].

Charges \ Monodromies	$U^{L\pm}(\lambda_{L\pm})$	$U^{R\pm}(\lambda_{R\pm})$
$SL(2, \mathbb{R})_L$ Yangian	$\infty$	0
exotic symmetry	0	$\infty$
local charges	$\pm 1$	$\pm 1$

Table 1: The charges and the expansion points of monodromy matrices.

## 5 Conclusion and Discussion

We have unveiled an *exotic* hidden symmetry in Schrödinger sigma models. This is an infinite-dimensional extension of  $q$ -deformed Poincaré algebra. We have argued the tower structure of the conserved charges by directly evaluating the Serre relations. The tower is generated in an *exotic* way and it looks like a tilted Yangian algebra. There are two representations of the tower structure for the exotic symmetry, depending on the choice of the level 0 charges, and the one is mapped to the other simply through the sign flipping of  $\sqrt{C}$ .

There are the two descriptions to describe the classical dynamics: 1) the left description based on  $SL(2, \mathbb{R})_L$  and 2) the right description based on  $U(1)_R$ . Then the monodromy matrices in the two descriptions have been shown to be gauge-equivalent via the relation between the spectral parameters. Moreover, we have given a simple Riemann sphere interpretation of this equivalence. It can be understood both as the remnant of the left-right duality of  $SL(2, \mathbb{R})$  principal chiral models and as a decoupling limit of the hybrid

integrable structure of warped  $\text{AdS}_3$  sigma models.

One may suspect that there might be an isomorphism between the infinite-dimensional algebra obtained here and the standard Yangian. However, at a glance, the resulting algebra looks different from the Yangian because it contains a  $q$ -deformed Poincaré algebra, which has a constant parameter, as the level 0 part. Nevertheless, there might be such an isomorphism because the gauge-equivalence of the monodromy matrices means that both the exotic symmetry and the Yangian are realized by expanding a single monodromy matrix.

At least so far, we are not sure for the mathematical formulation of the exotic symmetry itself. It is important to reveal the Hopf algebraic structure of this symmetry, especially the coproduct. We believe that there should be a well-defined mathematical structure for it, not depending on the sigma model action, like in the case of non-local charges in  $O(N)$  non-linear sigma models [14, 15] and Yangian [30].

It is nice to look for some applications of the exotic symmetry in the context of warped  $\text{AdS}_3/\text{dipole CFT}_2$  [39, 40]. This duality is proposed as a toy model of the Kerr/CFT correspondence [41]. It would also be interesting to consider the quantum integrability of Schrödinger sigma models, though we have confined ourselves to the classical integrability so far. In this direction it is worth trying to construct the Bethe ansatz by following the procedure in [42–45].

We hope that the exotic symmetry unveiled here would play a significant role in exploring a new aspect of integrable deformations of  $\text{AdS}/\text{CFT}$ .

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